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# Performance Lower Bounds in Stochastic Robust and Adaptive Control

Ji Feng Zhang and Le Yi Wang

***Abstract*—The fundamental issues of capability of robust and adaptive control in dealing with uncertainty are investigated in stochastic systems. It is revealed that to capture the intrinsic limitations of adaptive control, it is necessary to use $\sup_t$ types of transient and persistent performance, rather than $\limsup_t$ types which reflect only asymptotic behavior of a system. For clarity and technical tractability, a simple first-order linear time-varying system is employed as a vehicle to explore performance lower bounds of robust and adaptive control. Optimal performances of nominal, robust and adaptive control are explicitly derived and their implications are discussed in an information framework. An adaptive strategy is scrutinized for its achievable performance bounds. The results indicate that intimate interaction and inherent conflict between identification and control result in a certain performance lower bound which does not approach the nominal performance even when the system varies very slowly. Explicit lower bounds are obtained when disturbances are either normally or uniformly distributed.**

***Index Terms*—Adaptive control, performance lower bounds, robust control, Stochastic system, time-varying parameter, uncertainty.**

## I. INTRODUCTION

This note studies the long-standing and intricate questions: What is the inherent impact of interactions between identification and control in adaptation? Can adaptive control provide much larger capability in dealing with uncertainty? How can we quantify the impact of time variation on system robustness and achievable performance? Although there are many intuitions and research findings which provide guidelines in pursuing answers to these questions, clear and quantifiable conclusions on these questions are well known to be extremely difficult. Some of these technical difficulties are inherent: To understand essential capability and fundamental limitations of robust and adaptive control, one must obtain either optimal or at least lower performance bounds. In addition, these issues are mostly imminent only in a system's transient and persistent performance, in contrast to asymptotic performance. Knowing that derivations of upper bounds of asymptotic performance of a fixed adaptation algorithm have been painfully difficult, one can perceive the challenges involved in pursuing lower bounds of transient and persistent performance over all possible adaptive algorithms.

As a compromise, in this note we employ a first-order linear time-varying stochastic system as a vehicle to explore these issues. Optimal performances of nominal, robust and adaptive control are explicitly derived and their implications are discussed in an information framework. In particular, it is shown that when no information on parameter evolution is available, adaptation is not applicable and robust control, which is designed on the basis of prior information, can only provide very limited robustness against plant uncertainty. On the other hand, when plant parameters vary relatively slowly, adaptation can be employed to

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dramatically enhance robustness. An adaptive control strategy, which utilizes least-mean-squares identification, certainty equivalence principles, and optimal disturbance attenuation, is investigated for its achievable performance bounds. The results indicate that intimate interaction and inherent conflict between identification and control result in a certain performance lower bound which does not approach the nominal performance even when the system varies very slowly. This finding, somehow surprising, is a clear indication that information acquisition and processing are indeed inherently intertwined. Explicit and tighter lower bounds are obtained when disturbances are either normally or uniformly distributed.

Deriving optimal or lower bounds on system performance in the presence of uncertainty is one of the main thrusts in information and complexity based theory of feedback, identification and adaptation. This direction has been mainly pursued in deterministic worst-case frameworks. For an explorative exposure to the basic theory of identification and adaptive control, the reader is referred to the books [1]–[4], and the references therein. The deterministic counterpart of the results of this note is reported in [5].

## II. Problem Formulation

Consider a first-order system

$$y(t) = a(t)y(t-1) + u(t) + w(t), \qquad t \in \mathcal{N} \tag{1}$$

where $u(t)$, $y(t)$, $a(t)$ and $w(t)$ are system input, output, parameter and noise, respectively, and $\mathcal{N}$ is the set of nonnegative integers. $\{w(t),\, t \in \mathcal{N}\}$ is an independent random sequence with $Ew(t) = 0$ and $Ew^2(t) = \delta^2$. The system parameter $\{a(t),\, t \in \mathcal{N}\}$ is also a random sequence which is independent of the noise $\{w(t),\, t \in \mathcal{N}\}$. Further information on $a(t)$ will be specified later. For causality of control actions, $u(t)$ is limited to be a function of the data $\{u(0), \ldots, u(t-1), y(0), \ldots, y(t-1)\}$ and information on $a(t)$ available at $t-1$.

Our control goal is to minimize the effect of the disturbance $w$ on $y$. Namely, $Ey^2(t)$ is to be reduced. For reasons which will become clear shortly, we employ the transient and persistent performance $\sup_{t\ge 1} Ey^2(t)$, in contrast to the asymptotic performance $\limsup_{t\to\infty} Ey^2(t)$.

Control strategies can be devised on the basis of available information on the plant parameter $a(t)$. Depending on the available information on $a(t)$ and its utility in designing control $u(t)$, we may introduce distinctively the notion of nominal control, robust control and adaptive control as follows.

For a given $t > 0$, let $\mathcal{I}_\tau(a(t))$ denote the information on $a(t)$ available at time $\tau$, $\tau = 0, 1, \ldots, t-1$. In particular, $\mathcal{I}_0(a(t))$ is the prior information on $a(t)$.

4) If $\mathcal{I}_{t-1}(a(t))$ contains a single value $a_0(t)$, namely, no uncertainty on $a(t)$, then the corresponding control design and performance are called *nominal design and nominal performance*. While nominal performance does not have practical value, it is used in this note as a benchmark value to assess adaptive performance.
5) If the prior information $\mathcal{I}_0(a(t))$ cannot be further improved in the time interval $\tau = 1, 2, \ldots, t-1$, namely, $\mathcal{I}_0(a(t)) = \mathcal{I}_1(a(t)) = \cdots = \mathcal{I}_{t-1}(a(t))$, then $u(t)$ can only be designed on the basis of $\mathcal{I}_0(a(t))$, which will be called *robust control*.
6) If $\mathcal{I}_\tau(a(t))$, $\tau = 1, 2, \ldots, t-1$, improves $\mathcal{I}_0(a(t))$, then it is possible to design $u(t)$ based on the better information $\mathcal{I}_{t-1}(a(t))$. In this case, the design is called *adaptive design*.

These concepts will be made concrete and accurate in the subsequent sections. In each of these cases we will focus on deriving lower bounds on the optimal disturbance attenuation

$$\eta = \inf_{\{u(t)\}} \sup_{t\ge 1} Ey^2(t).$$

For a random process $\{x(t),\, t \ge 0\}$, the simplified notation $x = \{x(t)\} = \{x(t),\, t \ge 0\}$ will often be used.

## III. Nominal Performance

If $a(t)$ can be directly and accurately measured prior to designing the control action $u(t)$, then there is no uncertainty on the system parameters and the optimal control can be trivially obtained as $u(t) = -a(t)y(t-1)$. The corresponding optimal adaptive performance is

$$\eta = \delta^2.$$

While the nominal performance cannot be achieved in practice, due to inevitable measurement errors, this bound will serve as an important benchmark for understanding the inherent complexity of adaptive control.

## IV. Robust Control

Suppose that the prior information $\mathcal{I}_0(a(t))$ can be expressed as

$$a(t) = a_0(t) + v(t), \qquad \forall\, t \ge 1$$

where $a_0(t)$ is known *a priori* and deterministic. $\{v(t)\}$ is an independent random process with $Ev(t) = 0$, $Ev^2(t) = \varepsilon^2$, and is independent of $\{w(t)\}$. Due to the independence of $\{v(t)\}$ no additional information on $a(t)$ can be extracted from the values of $\{a(t-1), \ldots, a(0)\}$ or input/output observations $\{u(0), \ldots, u(t-1), y(0), \ldots, y(t-1)\}$ up to $t-1$. As a result, $\mathcal{I}_\tau(a(t)) = \mathcal{I}_0(a(t))$, $\tau = 1, 2, \ldots, t-1$. It follows that adaptation is not applicable and robust control becomes the only viable option.

*Theorem 1:* Suppose $\{v(t)\}$ is an independent random sequence, and independent of $\{w(t)\}$.

4) For any causal control sequence $\{u(t)\}$

$$Ey^2(t) \ge \varepsilon^2 Ey^2(t-1) + \delta^2.$$

Moreover the inequality becomes an equality when $u(t) = -a_0(t)y(t-1)$.
5) $Ey^2(t)$ is uniformly bounded if and only if $\varepsilon < 1$. When $\varepsilon < 1$, $y(0) = 0$ a.s., and $u(t) = -a_0(t)y(t-1)$

$$\eta = \frac{\delta^2}{1-\varepsilon^2}.$$

*Proof:*

4) Note that in this case, the data available to design $u(t)$ is only $\mathcal{D}_{t-1} = \{a_0(t), a_0(\tau), y(\tau), u(\tau), 0 \le \tau \le t-1\}$. This implies that $u(t) \in \mathcal{F}_{t-1}$, where $\mathcal{F}_t$ denotes the $\sigma$-algebra generated by data $\mathcal{D}_t$. Hence

$$Ev(t)y(t-1)w(t) = 0,$$
$$E[a_0(t)y(t-1) + u(t)]w(t) = 0$$
$$Ev(t)y(t-1)[a_0(t)y(t-1) + u(t)] = 0$$

and

$$\begin{aligned} Ey^2(t) &= E\{v(t)y(t-1) + (a_0(t)y(t-1) + u(t)) + w(t)\}^2 \\ &= E[v(t)y(t-1)]^2 \\ &\quad + E[a_0(t)y(t-1) + u(t)]^2 + E[w(t)]^2 \end{aligned}$$

$$
\begin{aligned}
&+2Ev(t)y(t-1)[a_0(t)y(t-1)+u(t)]\\
&+2Ev(t)y(t-1)w(t)\\
&+2E[a_0(t)y(t-1)+u(t)]w(t)\\
&=E[v(t)y(t-1)]^2+E[a_0(t)y(t-1)+u(t)]^2+\delta^2\\
&\geq \varepsilon^2 E[y(t-1)]^2+\delta^2.
\end{aligned}
$$

The inequality becomes an equality when $E[a_0(t)y(t-1)+u(t)]^2 = 0$. Hence, $u(t) = -a_0(t)y(t-1)$ is the optimal control.

5) Under the optimal control

$$Ey^2(t)=\varepsilon^2 E[y(t-1)]^2+\delta^2$$

is a dynamic system on $Ey^2(t)$ which is stable if and only if $\varepsilon < 1$. When $\varepsilon < 1$ and $y(0)=0$ a.s.

$$Ey^2(t)=\frac{\delta^2(1-\varepsilon^{2t})}{1-\varepsilon^2}, \qquad t=1,2,\ldots.$$

This implies that

$$\eta=\frac{\delta^2}{1-\varepsilon^2}.$$

□

From the necessary and sufficient condition $\varepsilon < 1$, it is clear that since robust control does not utilize any further information on the dynamics of $a(t)$ beyond $\mathcal{I}_0(a(t))$, it cannot deal with large uncertainties.

## V. Adaptive Control

Adaptation becomes applicable if further information about $a(t)$ can be obtained from the past data $\{a(\tau), y(\tau), u(\tau), 0 \leq \tau \leq t-1\}$. Here, we shall investigate two situations.

In the first situation, $a(t)$ evolves from $a(t-1)$ by

$$a(t)=a(t-1)+v(t). \tag{2}$$

$a(t-1)$ is accurately measured after $t-1$ but prior to $t$. $v$ has zero mean and variance $\gamma^2$. $v$ represents time-variation and $\gamma$ is a measure of variation rate.

Observe that the information $\mathcal{I}_\tau(a(t))$ can be expressed as

$$a(t)=a(\tau)+d(t,\tau), \qquad \tau=0,1,\ldots,t-1$$

with $d(t,\tau)=\sum_{i=\tau+1}^{t} v(i)$ and $E\,d^2(t,\tau)=(t-\tau)\gamma^2$. Clearly, the uncertainty decreases with $\tau$. In particular, the prior information $\mathcal{I}_0(a(t))$ is

$$a(0)+d(t,0)$$

with $E\,d^2(t,0)=t\gamma^2$. For large $t$, this represents very large uncertainty. By Theorem 1, robust control, which uses $\mathcal{I}_0(a(t))$ only, is not capable of dealing with such type of uncertainties. On the other hand, it will be shown that adaptation, which employs the newest information $\mathcal{I}_{t-1}(a(t))$, can provide satisfactory control. From this point of view, it is much more powerful than robust control in dealing with such uncertainties.

Furthermore, it will be revealed that the capability of adaptation is limited by the maximum allowable variation rate $\gamma$. This is one evidence that without further information on parameter evolution, adaptation can only deal with slowly varying systems.

In the second case, $a(t)$ still evolves from $a(t-1)$ by

$$a(t)=a(t-1)+v(t) \tag{3}$$

but no further assumption is made on $a(t-1)$. In other words, we are dealing with a random walk model. Only information about the parameter evolution is its variation rate $\gamma$. On the other hand, some information on $a(t-1)$ can be obtained from the data $y(t-1)$, $y(t-2)$, $u(t-1)$ via system identification.

### A. Gain Scheduling: Slowly Varying Systems with Noisy Parameter Measurements

Suppose that $a(t)$ and $a(t-1)$ are related by (2).

*Theorem 2:* Suppose $\{v(t)\}$ and $\{w(t)\}$ are independent random sequences and independent of each other, with zero mean and variances $\gamma^2$ and $\delta^2$, respectively.

4) For any causal control sequence $\{u(t)\}$

$$Ey^2(t)\geq \gamma^2 Ey^2(t-1)+\delta^2.$$

Moreover, the inequality becomes an equality when $u(t) = -a(t-1)y(t-1)$.

5) $Ey^2(t)$ is uniformly bounded if and only if $\gamma < 1$. When $\gamma < 1$, $y(0)=0$ a.s., and $u(t)=-a(t-1)y(t-1)$

$$\eta=\frac{\delta^2}{1-\gamma^2}.$$

*Proof:* Comparing

$$a(t)=a(t-1)+v(t)$$

with the case of robust control, it is apparent that it is equivalent to the information in robust control with $v(t)$ as the noise, which is zero mean, variance $\gamma$ and independent of $w$. As a result, Theorem 2 can be proved with similar arguments and derivations to those in the proof of Theorem 1. □

Theorem 2 demonstrates an interesting interpretation of time-variation. While it is intuitively understood that time variation of a system introduces additional uncertainty on the system, it is usually very difficult to quantify this perception. Due to the simple structure of the systems involved here, the uncertainty on time variation measured by the rate $\gamma$ contributes exactly to the total uncertainty which feedback control must tolerate.

### B. Slowly Varying Systems Without Direct Parameter Measurements

Now we study the case where the only information on $a(t)$ is given by (3) and its variation rate $\gamma$. In this case, it becomes mandatory that $a(t-1)$ be identified via input–output observations $y(t-1)$, $y(t-2)$, $u(t-1)$. As a result, control $u(t-1)$ must play dual roles of identification and control. Some fundamental issues arise: What are the inherent tradeoff between identification and control in this case? What is the best achievable performance? The following conditions are imposed at the outset to facilitate analysis.

*Assumption 1:* $\{w(t)\}$ and $\{v(t)\}$ are independent random sequences and independent of each other; $w(t)\neq 0$ a.s.; $Ew(t) = Ew^3(t)=Ev(t)=0$, $Ew^2(t)=\delta^2$, $\delta>0$, $\sup_{t\geq 0} Ew^4(t)<\infty$; and $Ev^2(t)=\gamma^2 < 1$.

*Assumption 2:* $y(0)=0$ a.s. $u(t)\in\mathcal{F}_{t-1}$, where $\mathcal{F}_t$ is the $\sigma$-algebra generated by $\{y(\tau), u(\tau), 0\leq\tau\leq t\}$.

*Theorem 3:* In addition to Assumptions 1–2, suppose that the estimate $\hat{a}(t-1)$ of $a(t-1)$ is given by the following least-mean-squares algorithm:

$$\hat{a}(t-1)=\operatorname*{argmin}_a E(y(t-1)-ay(t-2)-u(t-1))^2 \tag{4}$$

and the control $u(t)$ is designed based on the minimum variance cost function $Ey^2(t)$ and the certainty equivalence principle

$$u(t)=-\hat{a}(t-1)y(t-1). \tag{5}$$

Then

$$\sup_{t\geq 1} Ey(t)^2 \geq \frac{1+\sqrt{5-4\gamma^2}}{2(1-\gamma^2)}\,\delta^2. \tag{6}$$

*Proof:* By $w(t)\neq 0$ a.s. and $w(t)$ is independent of $a(t)y(t-1)+u(t)$, we have that $y(t)\neq 0$ a.s. for all $t\geq 1$. Thus, from (4), it follows:

$$\hat{a}(t-1)=\frac{y(t-1)-u(t-1)}{y(t-2)}, \text{ a.s.} \qquad \forall t\geq 3,$$

or equivalently

$$y(t-1)=\hat{a}(t-1)y(t-2)+u(t-1), \text{ a.s.} \qquad \forall t\geq 3$$

which, together with (1), gives $\tilde{a}(t-1)y(t-2)=-w(t-1)$, a.s, or equivalently

$$\tilde{a}(t-1)=-w(t-1)y(t-2)^{-1}, \text{ a.s.} \qquad \forall t\geq 3 \tag{7}$$

where $\tilde{a}(t-1)=a(t-1)-\hat{a}(t-1)$.

Observing that from (1)

$$y(t)=v(t)y(t-1)+a(t-1)y(t-1)+u(t)+w(t)$$

and from Assumptions 1–2

$$\begin{aligned} Ev(t)y(t-1)(a(t-1)y(t-1)+u(t))&=0,\\ Ev(t)y(t-1)w(t)&=0\\ E(a(t-1)y(t-1)+u(t))w(t)&=0. \end{aligned}$$

Therefore

$$\begin{aligned} Ey^2(t) &= Ev^2(t)y^2(t-1)\\ &\quad + E(a(t-1)y(t-1)+u(t))^2+\delta^2\\ &\quad + 2Ev(t)y(t-1)(a(t-1)y(t-1)+u(t))\\ &\quad + 2Ev(t)y(t-1)w(t)\\ &\quad + 2E(a(t-1)y(t-1)+u(t))w(t)\\ &= \gamma^2 Ey^2(t-1)+E(a(t-1)y(t-1)+u(t))^2+\delta^2\\ &= \gamma^2 Ey^2(t-1)+E(\tilde{a}(t-1)y(t-1))^2+\delta^2 \end{aligned} \tag{8}$$

where $\hat{a}(t-1)y(t-1)+u(t)=0$ has been used.

By (7)

$$\begin{aligned} &E(\tilde{a}(t-1)y(t-1))^2\\ &\quad = E[w(t-1)y(t-1)/y(t-2)]^2\\ &\quad = E[w(t-1)(a(t-1)y(t-2)+u(t-1)+w(t-1))/y(t-2)]^2\\ &\quad = Ew^2(t-1)(a(t-1)y(t-2)+u(t-1))^2/y^2(t-2)\\ &\qquad + 2Ew^3(t-1)(a(t-1)y(t-2)+u(t-1))/y^2(t-2)\\ &\qquad + Ew^4(t-1)/y^2(t-2). \end{aligned} \tag{9}$$

From Assumption 1 it follows that $Ew^3(t-1)=0$ and $w(t-1)$ is independent of $(a(t-1)y(t-2)+u(t-1))/y^2(t-2)$. This implies

$$\begin{aligned} &Ew^3(t-1)(a(t-1)y(t-2)+u(t-1))/y^2(t-2)\\ &\quad = Ew^3(t-1)E(a(t-1)y(t-2)+u(t-1))/y^2(t-2)\\ &\quad = 0. \end{aligned}$$

This and (9) imply that

$$\begin{aligned} &E(\tilde{a}(t-1)y(t-1))^2\\ &\quad = Ew^2(t-1)(a(t-1)y(t-2)+u(t-1))^2\\ &\qquad /y^2(t-2)+Ew^4(t-1)/y^2(t-2)\\ &\quad \geq Ew^4(t-1)/y^2(t-2)\\ &\quad = Ew^4(t-1)Ey^{-2}(t-2)\\ &\quad \geq \delta^4 Ey^{-2}(t-2)\geq \delta^4\left[Ey^2(t-2)\right]^{-1} \end{aligned} \tag{10}$$

where we have used $\delta^4=(Ew^2(t-1))^2\leq Ew^4(t-1)$ and the fact that for any random variable $x$, $1\leq (Ex^2)(Ex^{-2})$, or equivalently, $(Ex^2)^{-1}\leq Ex^{-2}$.

Substituting (10) into (8) we have

$$\begin{aligned} Ey^2(t) &\geq \gamma^2 Ey^2(t-1)\\ &\quad + \delta^4[Ey^2(t-2)]^{-1}+\delta^2, \quad \text{a.s.} \quad \forall t\geq 3. \end{aligned} \tag{11}$$

Let $y_\infty=\sup_{t\geq 1} Ey(t)^2$. Then from (11) it follows that:

$$\sup_{t\geq 2} Ey^2(t)\geq \gamma^2 Ey^2(t-1)+\delta^4 y_\infty^{-1}+\delta^2, \quad \text{a.s.} \quad \forall t\geq 3.$$

This in turn gives

$$\sup_{t\geq 2} Ey^2(t)\geq \gamma^2 \sup_{t\geq 2} Ey^2(t)+\delta^4 y_\infty^{-1}+\delta^2$$

or equivalently

$$\sup_{t\geq 2} Ey^2(t)\geq \frac{\delta^4 y_\infty^{-1}+\delta^2}{1-\gamma^2}. \tag{12}$$

If (6) were not true, i.e.,

$$y_\infty < \frac{1+\sqrt{5-4\gamma^2}}{2(1-\gamma^2)}\,\delta^2 \tag{13}$$

then, by (12), we would have

$$\begin{aligned} \sup_{t\geq 2} Ey^2(t) &\geq \frac{\delta^4 \dfrac{2(1-\gamma^2)}{(1+\sqrt{5-4\gamma^2})}\,\delta^{-2}+\delta^2}{1-\gamma^2}\\ &= \frac{1+\sqrt{5-4\gamma^2}}{2(1-\gamma^2)}\,\delta^2. \end{aligned}$$

This implies

$$y_\infty \geq \sup_{t\geq 2} Ey^2(t)\geq \frac{1+\sqrt{5-4\gamma^2}}{2(1-\gamma^2)}\,\delta^2.$$

This contradicts (13). Thus, (6) is true. □

*Remarks:*

4) Assumptions 1–2 are standard, which are satisfied by a broad class of $\{w(t)\}$ and $\{a(t)\}$. For instance, $\{w(t)\}$ can be i.i.d. random sequences with normal or uniform distributions. $\{a(t)\}$ can be constant ($\gamma = 0$); or $a(t) = \sum_{i=0}^{t}\xi(i)$ with $\{\xi(t)\}$ being an independent random sequence with $E\xi(t)=0$ and $E\xi^2(t)=\gamma^2<1$.

5) In adaptive control, the optimal performance is bounded below by $((1+\sqrt{5})/2)\,\delta^2$ and does not approach the nominal performance $\delta^2$, even when $\gamma\to 0$. This discontinuity reveals a fundamental performance limitation caused by interactions between identification and control.

The bounds in Theorem 3 can be further improved if additional information on noise distributions become available. These are presented in the following.

*Corollary 1:*

4) In addition to the hypothesis of Theorem 3, if $\{w(t)\}$ is i.i.d. and normally distributed, then

$$\eta \geq \frac{1+\sqrt{13-12\gamma^2}}{2(1-\gamma^2)}\,\delta^2 > 2.3\delta^2, \qquad \text{for all } \gamma \in [0,\,1). \tag{14}$$

5) In addition to the hypothesis of Theorem 3, if $\{w(t)\}$ is i.i.d. and uniformly distributed, then

$$\eta \geq \frac{1+\sqrt{8.2-7.2\gamma^2}}{2(1-\gamma^2)}\,\delta^2 > 1.9\delta^2, \qquad \text{for all } \gamma \in [0,\,1). \tag{15}$$

*Proof:*

4) In the case where $\{w(t)\}$ is i.i.d. and normally distributed with $Ew(t) = 0$ and $Ew^2(t) = \delta^2$, the corresponding density function is

$$f_w(x) = \frac{1}{\delta\sqrt{2\pi}}\,\exp\left\{-\frac{x^2}{2\delta^2}\right\}.$$

Hence, we have $Ew^4(t-1) = 3\delta^4$. Applying this, instead of $Ew^4(t-1) \geq \delta^4$, to (10), we get

$$\sup_{t\geq 2} Ey^2(t) \geq \frac{3\delta^4 y_\infty^{-1} + \delta^2}{1-\gamma^2}$$

from which and

$$\frac{1+\sqrt{13-12\gamma^2}}{2(1-\gamma^2)} \geq \frac{1+\sqrt{13}}{2} > 2.3$$

for all $\gamma \in [0,\,1)$ we can prove (14) by contradiction.

5) In the case where $\{w(t)\}$ is i.i.d. and uniformly distributed with $Ew(t) = 0$ and $Ew^2(t) = \delta^2$, the corresponding density function is

$$f_w(x) = \begin{cases} \dfrac{1}{2\sqrt{3}\delta}, & \text{if } x \in \left[-\sqrt{3}\delta,\, \sqrt{3}\delta\right] \\ 0, & \text{otherwise.} \end{cases}$$

Hence, we have $Ew^4(t-1) = 1.8\delta^4$. Applying this, instead of $Ew^4(t-1) \geq \delta^4$, to (10), we get

$$\sup_{t\geq 2} Ey^2(t) \geq \frac{1.8\delta^4 y_\infty^{-1} + \delta^2}{1-\gamma^2}$$

from which and

$$\frac{1+\sqrt{8.2-7.2\gamma^2}}{2(1-\gamma^2)} \geq \frac{1+\sqrt{8.2}}{2} > 1.9$$

for all $\gamma \in [0,\,1)$ we can prove (15) by contradiction.

□

## VI. Conclusion

Performance lower bounds of robust and adaptive control are of essential importance in our understanding of the capability and limitations of robust and adaptive control. The key observations in this pursuit are: 1)without using online information, robust control can provide only very limited capability in dealing with uncertainty; 2) time variation of a system introduces additional uncertainty and further reduces the power of robustness against other uncertainties; 3) employing additional information on the dynamics of system parameters, adaptation can dramatically enhance a system's robustness; 4)interaction between identification and control fundamentally limits achievable performance of an adaptive control strategy. By using a simple first-order linear time-varying system as a platform, we are able to explore these issues rigorously, derive exact performance limits and demonstrate these ideas quantitatively. Extension of the findings of this paper to more complicated systems is currently under investigation.

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# Controlled Invariance and Feedback Laws

P. d'Alessandro and E. De Santis

***Abstract*—We consider controlled invariance for cones, translated cones, polyhedra and various special polyhedral structures. For any polyhedron, if controlled invariance occurs, then all and nothing but the admissible controls can be obtained by an inequative feedback controller. For each special polyhedral structure we compare this feedback controller against piecewise affine, linear and affine feedback controller.**

***Index Terms*—Discrete-time systems, state feedback.**

## I. Introduction

We consider linear dynamic time-invariant discrete time systems, which are described by a recursive equation of the form

$$x(t+1) = Ax(t) + Bu(t) \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^p$, and $A$ and $B$ are matrices of consistent dimensions. The equation is understood to hold for a given $t_0$—a parameter that can be any integer and represents the initial time—initial state $x(t_0)$ and $t > t_0$. The state $x(t)$ is recursively determined by the equation for any $t > t_0$. Usually, in view of time-invariance, we will assume $t_0 = 0$ without restriction of generality. A set $H$ in $R^n$ is said to be controlled invariant under the system if for any $x \in H$ $\exists u$ such that $Ax + Bu \in H$. Note that if we denote by $S$ the set

$$S = \{x \colon \exists u,\, Ax + Bu \in H\} \tag{2}$$

then invariance prevails if and only if $H \subset S$ (throughout the note the symbol $\subset$ has the same meaning as the symbol $\subseteq$). This is in turn equivalent to the existence of a solution to the problem: for any initial state $x(t_0) \in H$, there exists an input function $u(\cdot)$ defined on

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